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A Frequency Transformation for Commensurate Transmission-Line Networks

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Abstract—**The frequency transformation $W=1/S$, where $S=\tanh(\gamma L)$, is investigated for commensurate transmission-line networks consisting of stubs, resistors, ideal transformers, and unit elements. This transformation takes transmission-line transformers into transmission-line lowpass filters and vice versa, lowpass (or bandstop) filters into highpass (or bandpass) filters and vice versa, and elliptic-function bandstop filters into elliptic-function bandpass filters and vice versa. The practicality of the transformation lies in the fact that element values of the transformed network are easily related to the corresponding element values of the original network. The transformation is useful because it provides an alternative viewpoint for synthesis, and because it reduces the number of tables of designs needed for various filter types. Several examples of designs using the transformation are given. One design is an unusual narrowband 3-dB directional coupler.**

I. Introduction

FREQUENCY transformations are commonly used in lumped-element network theory to convert a given filter network into a related filter network. For example, an often used frequency transformation is [1], [2]

$$s' \rightarrow As, \tag{1}$$

where the symbol $\rightarrow$ stands for "is replaced by," A is a constant, the primed variable is that of the original network, and the unprimed variable is that of the transformed network.[1] Transformation (1) is used to scale the bandwidth of the existing network to another preferred value. Other commonly used frequency transformations in lumped filter theory are [1], [2]

$$s' \rightarrow A/s \quad \text{(lowpass to highpass transformation)} \tag{2}$$

$$s' \rightarrow w\left(\frac{s}{\omega_0}\right) + \left(\frac{\omega_0}{s}\right) \quad \text{(lowpass to bandpass transformation)} \tag{3}$$

$$s' \rightarrow \frac{1}{w\left(\frac{s}{\omega_0}\right) + \left(\frac{\omega_0}{s}\right)} \quad \text{(lowpass to bandstop transformation).} \tag{4}$$

It is emphasized that in all cases the usefulness of these transformations lies in the fact that their effects on the responses of the network are *easily* related to changes in the element values of the network. Because such frequency transformations are available, a given lowpass filter may function as a prototype for a number of different types of filters, obviating the compilation of a multitude of designs for lowpass, highpass, bandpass, and bandstop filters.

Analogous transformations would be equally useful for commensurate transmission-line networks, if they could be developed. For the special class of commensurate transmission-line networks consisting of open- and short-circuited stubs, ideal transformers, and resistors, but without unit elements [18] (i.e., quarter-wavelength lines), transformations (1) through (4) can indeed be used. In most cases, however, realization of commensurate transmission-line networks without unit elements is impractical or impossible. Unfortunately, in the more general case of commensurate transmission-line networks, consisting of open- and short-circuited stubs, ideal transformers, resistors, and unit ele-

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[1] Throughout this paper we shall use primed variables to represent parameters of the original network and unprimed variables for those in the tranformed network.

ments, the usual frequency transformations of lumped-element network theory cannot be used. The fundamental reason for this is that for transmission-line networks with unit elements, the effects of the transformations on the network responses are *not* easily (if at all) related to changes in the element values of the network. However, an exception to this statement is the frequency transformation[2]

$$S' \rightarrow 1/S. \tag{5}$$

It has been found that for this transformation the effects on the network elements can be relatively easily accounted for. This transformation corresponds to (2), with $A=1$, and possesses corresponding properties. However, its effects on the parameters of transmission-line networks can be substantially different from the effects on the parameters of lumped-element networks because of the existence of unit elements. It is easily shown that transformation (5) takes transmission-line filters into transmission-line transformers and vice versa, lowpass distributed filters into highpass distributed filters and vice versa, elliptic-function bandstop distributed filters into elliptic-function bandpass distributed filters and vice versa. Sections II through IV investigate the properties of transformation (5) and describe the relationships between the original and the transformed networks. Section V explores possible applications to which the transformation may be put.

II. Theory

The general analytical properties of the complex transformation

$$S' \rightarrow 1/S = W, \tag{6}$$

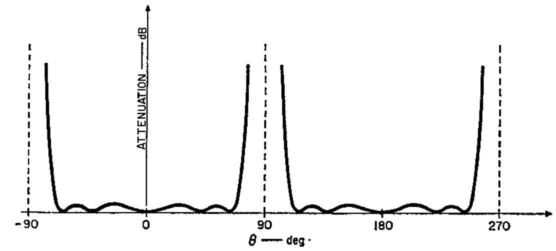
which represents the inversion of the unit circle in the complex plane, are not of particular interest in the present case. The interested reader may find these details in various references [4], [5]. Certain specific properties, however, should be pointed out. On the real frequency axis, the variable

$$W = 1/S = -j \cot \theta \tag{7}$$

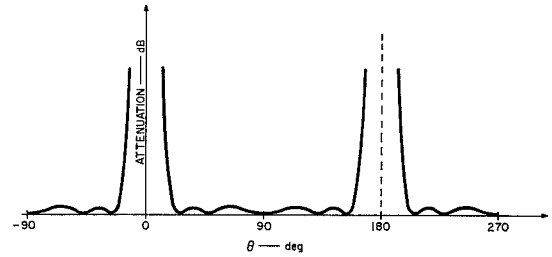
satisfies

$$-j \cot \theta = j \tan (\theta - 90), \tag{8}$$

where θ is the electrical length of the commensurate transmission lines and $j=\sqrt{-1}$. Thus transformation (6) is equivalent to shifting the origin 90° to the right for all network functions. The result of the translation can be seen to be, for example, that response functions of lowpass (or bandstop) networks become response functions of highpass (or bandpass) networks. This is illustrated by the insertion-loss functions shown in Figs. 1(a) and 1(b). Similarly, the response functions of stepped-impedance filters are transformed into the response functions of stepped-impedance transformers; those of short-step transformers [6] are transformed into the response functions of different short-step transformers; and those of bandstop elliptic-function filters [7]–[9] are transformed into the response functions of bandpass, elliptic-function filters.



(a). Lowpass (or bandstop) attenuation response.



(b). Highpass (or bandpass) attenuation response obtained by transforming the response of Fig. 1(a) by the mapping $S'=1/S$.

Fig. 1.

Note in Fig. 1 that a narrowband bandstop filter response transforms into a wideband bandpass filter response. Similar results are obtained for quarter-wavelength transformers and elliptic-function filters. The transformed bandwidths of short-step transformers [6], on the other hand, behave differently. The general relationships between bandwidths of original and transformed networks are derived later in Section III; for the present, let us consider the general relationships between the electrical parameters of the original and the transformed networks.

A. Networks of Only Commensurate Unit Elements

We consider first networks consisting of only unit elements, such as those represented in Fig. 2. Such networks may represent stepped-impedance transformers, lowpass filters, and prototype networks for directional couplers and other filter types [10], [11]. Let the original network be referred to as the S'-plane network and the transformed network as the W-plane network. Also, let impedances in the S' plane be primed and those in the W plane be unprimed.

Next, let an S'-plane network of unit elements be represented by the drawing of Fig. 3(a), so that the first unit element is placed in evidence. The corresponding W-plane network, yet to be determined, is similarly shown in Fig. 3(b). The input impedance for the S'-plane network[3] is

$$\bar{Z}_{\text{in}}(S') = Z_1' \left\{ \frac{\bar{Z}_L(S') + S'Z_1'}{Z_1' + S'\bar{Z}_L(S')} \right\}. \tag{9}$$

[2] For commensurate transmission-line networks, the variable S' represents $\tanh(\gamma L)$, where γ is the complex propagation constant and L is the commensurate length of the transmission lines [3].

[3] Input impedances (admittances) in the S' plane will be denoted with a bar. Input impedances (admittances) in the W plane will be denoted without a bar.

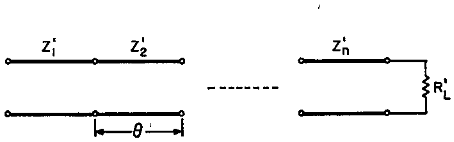
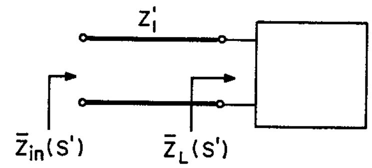
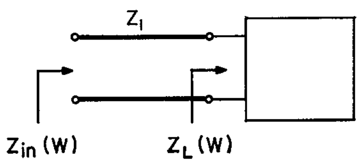


Fig. 2. A transmission-line network of commensurate unit elements,



(a). General network of commensurate unit elements in the S' plane. with the first unit element in evidence.



(b). General network of commensurate unit elements in the W plane, with the first unit element in evidence.

Fig. 3.

By (6), the impedance in the W plane is the same function Z_{in}, but in the variable W; hence

$$Z_{\text{in}}(W) = Z_1' \left\{ \frac{\bar{Z}_L(W) + WZ_1'}{Z_1' + W\bar{Z}_L(W)} \right\} = \bar{Z}_{\text{in}}(S')\,|_{S'=W}. \quad (10)$$

However, from first principles we know that the network of Fig. 3(b) can also be represented by the equation

$$Z_{\text{in}}(W) = Z_1 \frac{WZ_L(W) + Z_1}{WZ_1 + Z_L(W)}, \quad (11)$$

where the impedance of the unit element Z_1 and the impedance $Z_L(W)$ are yet to be determined. Using Richard's theorem [3],

$$Z_1' = \bar{Z}_{\text{in}}(S')\,|_{S'=1} = \bar{Z}_{\text{in}}(1) \quad (12)$$

and

$$Z_1 = Z_{\text{in}}(W)\,|_{W=1} = \bar{Z}_{\text{in}}(1); \quad (13)$$

thus

$$Z_1 = Z_1'. \quad (14)$$

Next, using Richard's reduction procedure [3] for removing unit elements,[4] we find that

$$\bar{Z}_L(S') = Z_1' \left\{ \frac{S'Z_1' - \bar{Z}_{\text{in}}(S')}{S'\bar{Z}_{\text{in}}(S') - Z_1'} \right\} = Z_1'\bar{G}(S'), \quad (15)$$

[4] A factor $1-(S')^2$ cancels in the numerator and denominator of (15) but is not shown here in order to keep the presentation simple. A similar procedure applies for a factor $1-W^2$ in (16).

where the definition of $\bar{G}(S')$ is clear from (15). Similarly, from (11)

$$Z_L(W) = Z_1 \frac{Z_1 - WZ_{\text{in}}(W)}{Z_{\text{in}}(W) - WZ_1}. \quad (16)$$

However, since $Z_1 = Z_1'$, (16) is equivalent to

$$Z_L(W) = Z_1'/\bar{G}(W), \quad (17)$$

where the function $\bar{G}$ in (17) is the same as that in (15) with S' replaced by W. Combining (15) and (17) gives the result

$$Z_L(W) \equiv (Z_1')^2/\bar{Z}_L(S')\,|_{S'=W}. \quad (18)$$

The identity symbol in (18) is used to convey the meaning that the functions on the left and right are identical but denoted in different variables. In words, (18) states that the impedance function remaining in the W plane after removing the first unit element is mathematically the same as the admittance function remaining in the S' plane scaled by the factor $(Z_1')^2$.

Next, steps (12) and (13) are repeated, with $\bar{Z}_{\text{in}}(S')$ replaced by $\bar{Z}_L(S')$ and $Z_{\text{in}}(W)$ replaced by $Z_L(W)=(Z_1')^2/\bar{Z}_L(W)$. We obtain for the second unit element in the S' plane,

$$Z_2' = \bar{Z}_L(S')\,|_{S'=1} \quad (19)$$

and for the second unit element in the W plane,

$$Z_2 = Z_L(W)\,|_{W=1} = (Z_1')^2/\bar{Z}_L(W)_{W=1} = (Z_1')^2 Y_2'. \quad (20)$$

Again using Richards' reduction procedure, the remaining impedance in the W plane is determined to be

$$Z_{L_2}(W) \equiv (Z_1')^2(Y_2')^2\bar{Z}_{L_2}(S')\,|_{S'=W}. \quad (21)$$

Thus, at the end of the second cycle of determining the impedances of the W-plane network, the impedance function in the W plane has returned to its original form [i.e., $Z_{\text{in}}(W) \equiv \bar{Z}_{\text{in}}(S')$] except for a scale factor. A continuation of the previous procedure gives, for the impedance in the W plane, alternately $\bar{Y}(S')$ and $\bar{Z}(S')$, scaled by the appropriate factors. The general relationships between corresponding impedances in the S' and W planes are thus

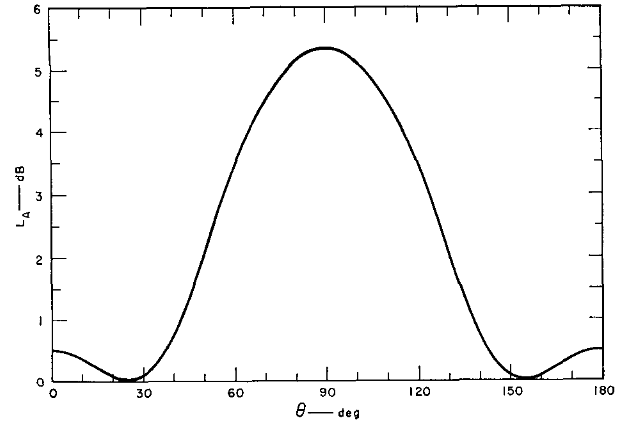
$$Z_i = [(Z_1')(Y_2')(Z_3') \cdots (Y_{i-1}')]^2 Z_i' \quad \text{for } i \text{ odd}, \quad (22)$$

and

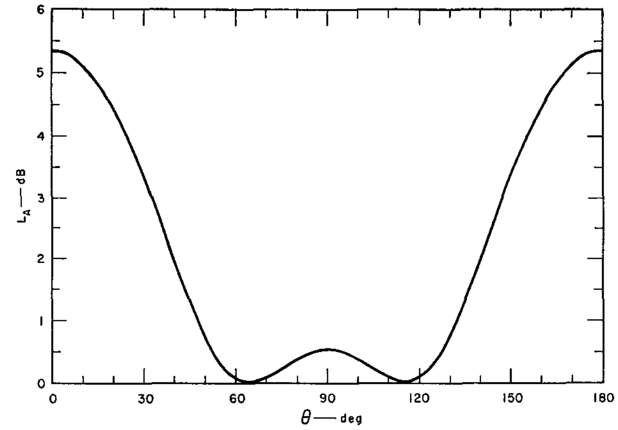
$$Z_i = [(Z_1')(Y_2')(Z_3') \cdots (Z_{i-1}')]^2 Y_i' \quad \text{for } i \text{ even}. \quad (23)$$

Equations (22) and (23) are equivalent to formulas for designing half-wave filters from quarter-wave prototype transformers [12]. We see from the previous discussion, however, that these equations are completely general. They apply to any arbitrary cascade of commensurate transmission lines, with the consequence that the network responses in $j \tan \theta$ are replaced by the same responses in $-j \cot \theta$.

To illustrate an application of the foregoing theory, consider the following example of the transformation of a two-section short-step transformer [6] of the following specifications:



(a). Attenuation response of two-section short-step transformer from the tables of Matthaei.



(b). Attenuation response of the transformed short-step transformer under the transformation $S'=1/S$.

Fig. 4.

Fractional bandwidth	w'	$=1.2$
Termination ratio	r	$=2:1$
Passband ripple	L_{A_r}'	$=0.39$ dB
	Z_1'	$=2.1956$
	Z_2'	$=0.91091$
Termination	R_L	$=2.0.$

The attenuation response of this transformer is given in Fig. 4(a), and the transformed attenuation response in the variable $1/S$ is given in Fig. 4(b). The response shown in Fig. 4(b) implies that the short-step transformers go into other short-step transformers under the transformation $S'\to 1/S$. Thus, in some cases, the transformation (6) may be useful in extending Matthaei's tables, although no statement can be made as to the generality of this particular application.

The transformed impedances in the W plane are calculated from (22) and (23), giving

$$Z_1 = 2.1956$$

$$Z_2 = (Z_1')^2 Y_2' = 5.2921$$

$$R_L = [(Z_1')(Y_2')]^2 R_L' = 11.619.$$

For this example the new termination ratio is seen to be 11.619; and the new 0.39-dB fractional bandwidth, determined from the data of Fig. 4(b), has been reduced to the value $w=0.403$. Thus the resulting network is indeed another short-step transformer.[5] To confirm these results, the response of the W-plane network was calculated with a digital computer and was found to agree with the response given in Fig. 4(b).

B. Distributed Networks of Stubs, Unit Elements, Ideal Transformers, and Resistors

The previous theory, developed for cascaded commensurate unit elements, is extended in this section to networks consisting of stubs, unit elements, ideal transformers, and resistors. The technique for developing the transformed network from the original network follows closely the concepts presented in Section II-A. In order to facilitate the description of the method, however, it will be useful to digress momentarily to examine the mathematical form of the immittances of stubs in the variable W.

Four diagrams of open- and short-circuited stubs, in series and in shunt with general networks, in both the S' and W planes, are given in Fig. 5. By referring to this figure during a network transformation problem, one can quickly determine the type of stub called for in either the S' or W plane. For example, assume that in a filter transformation problem, the impedance function

$$Z_{\text{in}}(W) = 25.6W + Z(W)$$

occurs. Reference to Fig. 5(b) shows immediately that the network is an open-circuited stub of characteristic impedance 25.6 ohms in series with a residual network having input impedance $Z(W)$.

The method of transforming a given network in the S' plane into another network in the W plane will be explained by means of a worked example. After the example has been given, certain general statements will be made that will enable a designer to transform one network into another without recourse to most of the mathematics presented during the description of the worked example. The example is the transformation of a five-resonator bandstop filter, having fractional bandwidth $w'=1.00$ and 0.1-dB Chebyshev ripple in its passband, into a bandpass filter. The bandstop filter is shown schematically in Fig. 6 with $n=5$. Normalized values for network admittances h are given in reference [13]. The transformed network is developed in the following way.

The input admittance of the bandstop filter seen from the left side of Fig. 6 is

$$\overline{Y}_{\text{in}}(S') = h_1 S' + \overline{Y}_1(S'), \tag{24}$$

where $\overline{Y}_1(S')$ is the admittance remaining to the right of the first stub, Y_1'.[6] Therefore, at the corresponding reference

[5] Because of the small termination ratio of the original transformer (i.e., 2:1), the transformed network will have only a 0.5-dB insertion loss at $\theta=90°$. Thus for this case the transformed network also represents a conventional transformer with termination ratio $r=11.62$, 0.5-dB equal-ripple response, and fractional bandwidth 1.64. However, this situation occurs only when the original short-step transformer consists of two sections and small termination ratio.

[6] For this example, input admitances in the S' plane will be denoted with a bar. Input admittances in the W plane will be without a bar.

plane, the transformed network in the W plane has the admittance [set $S' = W$ in (24)]

$$Y_{\text{in}}(W) = h_1 W + \bar{Y}_1(W). \tag{25}$$

Reference to Fig. 5(c) shows that this admittance may be represented by a short-circuited stub of characteristic admittance[7]

$$H_1 = h_1 \tag{26}$$

in shunt with a residual network of input admittance $\bar{Y}_1(W)$. The input admittance in the W plane, $Y_1(W)$, evidently satisfies

$$Y_1(W) \equiv \bar{Y}_1(S') \,|_{S'=W} \tag{27}$$

where again, the identity sign is used to emphasize the fact that the functions on the right and left of (27) are identical.

Now $\bar{Y}_1(S')$ is the admittance seen when looking to the right of the first stub of the bandstop filter. This admittance is a unit element of characteristic admittance h_{12} terminated in a load which will be designated as $\bar{Y}_2(S')$. Since $Y_1(W) \equiv \bar{Y}_1(S')$, in the W plane we also have a unit element terminated in a residual network. At this point in the development of the transformed network, the situation is exactly as that described in Section II-A. Hence, the unit element in the W plane is evaluated using the procedures in Section II-A, giving

$$H_{12} \equiv h_{12}. \tag{28}$$

Also, by the methods described in Section II-A [in particular (18)], the input impedance of the residual network in the W plane satisfies

$$Z_2(W) \equiv \frac{1}{h_{12}^2 \bar{Z}_2(S')} \Big|_{S'=W}$$

or

$$Z_2(W) \equiv \bar{Y}_2(S')/h_{12}^2 \,|_{S'=W}. \tag{29}$$

In words, (29) states that the input *impedance* of the remaining network in the W plane is mathematically equivalent to the input *admittance* of the remaining network in the S' plane, scaled by the factor $1/h_{12}^2$, and with S'replaced by W.

The development of the transformed network continues from (29). The input admittance in the S'-plane network is seen (from Fig. 6) to be an open-circuited stub in shunt with a residual network, designated by $\bar{Y}_3(S')$. Hence

$$\bar{Y}_2(S') = h_2 S' + \bar{Y}_3(S'). \tag{30}$$

Thus from (29) we have

$$Z_2(W) = \frac{h_2}{h_{12}^2} W + \frac{\bar{Y}_3(W)}{h_{12}^2} \cdot \tag{31}$$

Reference to Fig. 5(b) shows that $Z_2(W)$ may be represented by an open-circuited stub in series with a residual network having an input impedance, designated as $Z_3(W)$, which satisfies

$$Z_3(W) = \frac{\bar{Y}_3(W)}{h_{12}^2} \cdot \tag{32}$$

The characteristic admittance of the open-circuited stub is

$$H_2 = \frac{h_{12}^2}{h_2} \cdot \tag{33}$$

Further development of the transformed network continues along similar lines. The admittance $\bar{Y}_3(S')$ is seen to be that of a unit element of characteristic admittance h_{23}, terminated in a residual network designated as $\bar{Y}_4(S')$. Therefore, by (32), in the W plane the corresponding network may also be represented by a unit element terminated in a residual network. The evaluation of the unit element is

$$\frac{1}{H_{23}} = Z_3(W)\Big|_{W=1} = \frac{\bar{Y}_3(W)}{h_{12}^2}\Big|_{W=1} = \frac{h_{23}}{h_{12}^2} \cdot \tag{34}$$

By the arguments put forth in Section II-A, the residual network in the W plane, having input impedance $Z_4(W)$, satisfies

$$Z_4(W) \equiv \frac{h_{23}^2 \bar{Z}_4(S')}{h_{12}^2} \Big|_{S'=W} \cdot \tag{35}$$

In words, (35) states that the input impedance of the residual

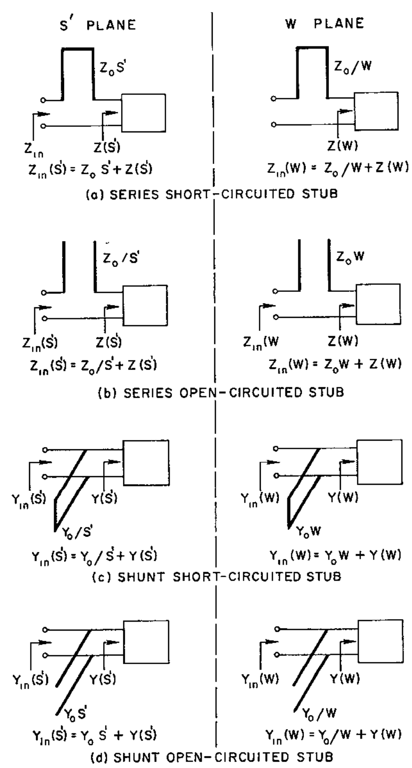


Fig. 5. Four diagrams of open- and short-circuited stubs and their mathematical forms in the S' plane and W plane.

[7] The characteristic admittances in the W plane will be denoted by H.

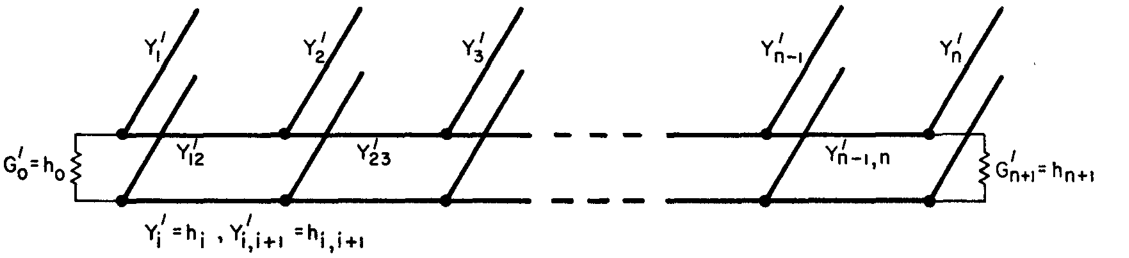


Fig. 6. Schematic drawing of transmission-line bandstop filter.

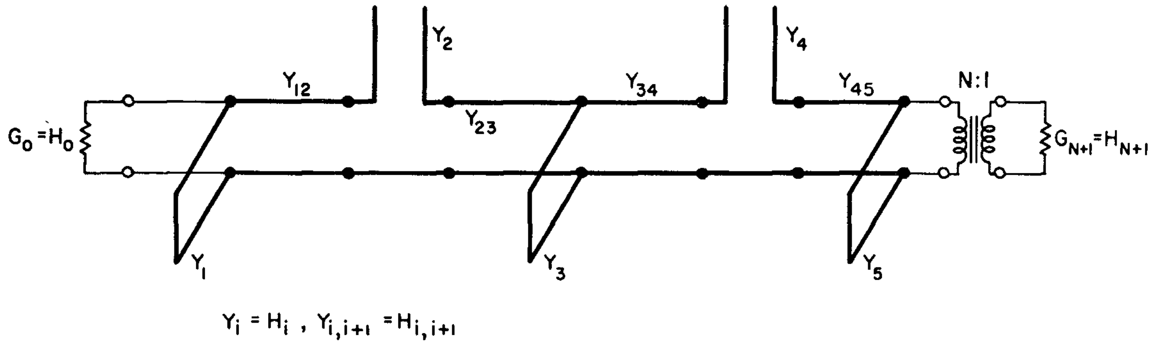


Fig. 7. Transmission-line bandpass filter developed from a bandstop filter using the frequency transformation $S'=1/S$.

network in the W plane is mathematically equivalent to the input impedance of the residual network in the S' plane, scaled by the factor h_{23}^2/h_{12}^2, and with S' replaced by W.

At this stage in the development of the transformed network, the mathematics of the synthesis method has returned to the starting point (24), except for a scaling factor h_{23}^2/h_{12}^2. The development of the remainder of the network follows the same cycle that has just been described. The procedure is carried on by inverting (35) to obtain the input admittance and following the steps beginning with (24).

A schematic of the final transformed network is given in Fig. 7. Note that an ideal transformer is required at the right side of the network. This is because, after development of the last stub in the transformed network, the residual resistance satisfies

$$R_L(W) = h_{12}^{-2}h_{23}^{2}h_{34}^{-2}h_{45}^{2}R_L(S')\,\big|_{S'=W} \qquad (36)$$

where $R_L(S')=1/h_{n+1}$. Hence the coefficient of $R_L(S')$ in (36) corresponds to an ideal transformer, as shown in Fig. 7, with turns ratio

$$N = h_{12}^{-1}h_{23}h_{34}^{-1}h_{45}. \qquad (37)$$

In terms of the normalized admittance parameters of the original network, the normalized admittance parameters of the transformed network are

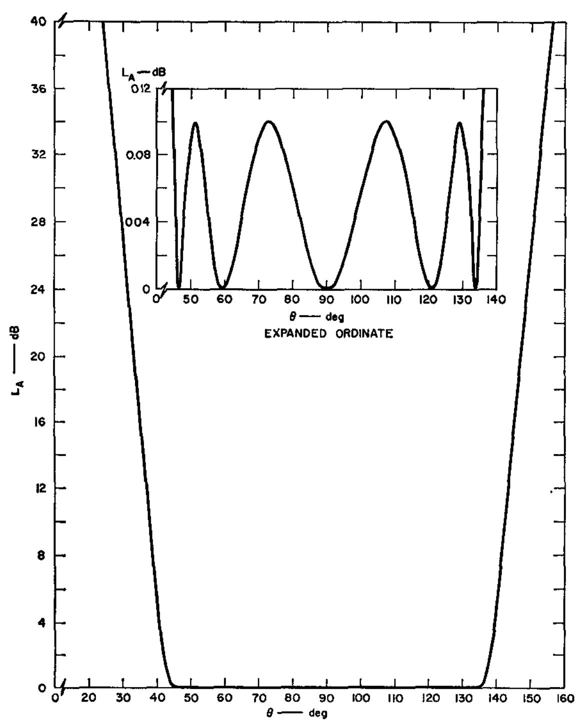


Fig. 8. Attenuation response of the transformed bandstop network given as an example in the text.

$$
\begin{aligned}
H_1 &= h_1 \\
H_{12} &= h_{12} \\
H_2 &= h_{12}^2/h_2 \\
H_{23} &= h_{12}^2/h_{23} \\
H_3 &= h_{12}^2 h_{23}^{-2} h_3 \\
H_{34} &= h_{12}^2 h_{23}^{-2} h_{34} \\
H_4 &= h_{12}^2 h_{23}^{-2} h_{34}^2/h_4 \\
H_{45} &= h_{12}^2 h_{23}^{-2} h_{34}^2 h_{45}^{-1} \\
H_5 &= h_{12}^2 h_{23}^{-2} h_{34}^2 h_{45}^{-2} h_5 \\
N^2 &= h_{12}^{-2} h_{23}^2 h_{34}^{-2} h_{45}^2 \\
H_6 &= h_6.
\end{aligned} \tag{38}
$$

For the particular example just illustrated, the h parameters of the original network satisfy certain symmetry conditions [13] which, when taken into account, reduce (38) to

$$
\begin{aligned}
H_1 &= h_1 = H_5 \\
H_{12} &= h_{12} = H_{45} \\
H_2 &= h_{12}^2/h_2 = H_4 \\
H_{23} &= h_{12}^2/h_{23} = H_{34} \\
H_3 &= h_{12}^2 h_{23}^{-2} h_3 \\
N &= 1 \\
H_6 &= h_6.
\end{aligned} \tag{39}
$$

In order to verify the results given in (39), the network given in Fig. 7 was analyzed on a computer using the relationships in (39) and the numerical data from reference [13]. The calculated insertion-loss function is shown in Fig. 8, and it has been verified that this is the same insertion-loss function of the original bandstop filter, with S' replaced by W.

A brief review of the previously described transformation procedure reveals a pattern that makes it unnecessary, in most cases, to perform any of the previously outlined mathematical steps. With a little practice, one can write down the parameters of the transformed network by inspection. Note that the requirement to renormalize and invert the residual network immittance arises only after the removal of each unit element. The removal of stubs, ideal transformers, and resistors leaves intact the form of the residual network. Thus the transformation of a given network by the mapping $S' \rightarrow W$ can be accomplished as follows.

Unit elements in the transformed network are developed according to the rules given in Section II-A, without regard to the presence of stubs. The development of stubs in the transformed network is divided into two cases:

1) If an odd number of unit elements has been developed, *shunt* open- or short-circuited stubs are transformed into *series* open- or short-circuited stubs (and vice versa), respectively, with appropriate immittance scaling.
2) If an even number of unit elements has been developed, *shunt* open- or short-circuited stubs are transformed into *shunt* short- or open-circuited stubs, respectively, with appropriate immittance scaling; and *series* open- or short-circuited stubs are transformed into *series* short- or open-circuited stubs, respectively, with appropriate immittance scaling.

The "appropriate immittance scaling" referred to in 1) and 2) requires the factor (in terms of impedance)

$$[Z_1' Y_2' Z_3' \cdots Z_i']^2 \qquad \text{for } i \text{ odd,} \tag{40}$$

and

$$[Z_1' Y_2' Z_3' \cdots Y_i']^2 \qquad \text{for } i \text{ even,} \tag{41}$$

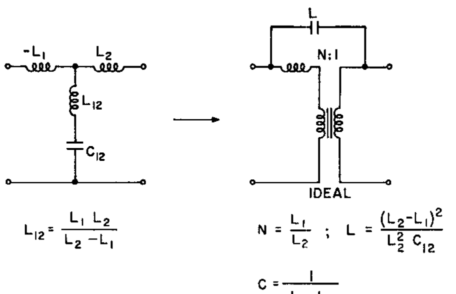
where Z_j' and Y_j' are the impedance and admittance, respectively, of the jth unit element of the S'-plane network.

C. *Application to Distributed Elliptic-Function Filters*

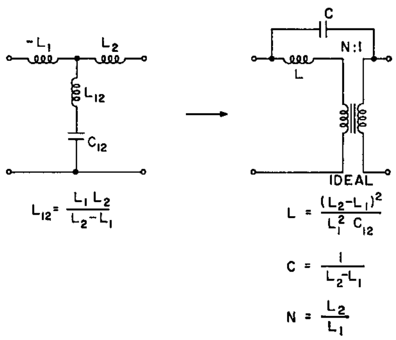
The theory presented in Sections II-A and II-B may also be applied to elliptic-function filters [7]–[9]. In the general case, elliptic-function filters may contain unit elements, shunt and series open- and short-circuited stubs, distributed LC shunt and series resonant and antiresonant sections, and distributed nondegenerate Brune sections.[8] Networks of unit elements and stubs have been discussed in Section II-B. Where circuit elements of the network are LC series and shunt resonant and antiresonant circuits, the extension of the theory is straightforward. For example, if an even number of unit elements has been developed in the transformed network, a shunt, LC series resonant circuit will be transformed into another shunt, LC series resonant circuit with the impedance level appropriately scaled. On the other hand, if an odd number of unit elements has been developed in the transformed network, a shunt, LC series resonant circuit will be transformed into a series, antiresonant circuit with the impedance level appropriately scaled. Extension to other types of LC circuits is obvious.

Application of the theory to Brune sections reveals that a Brune section transforms into another Brune section. Again, there are two cases to consider, depending on whether an even or odd number of unit elements has been developed in the transformed network. The transformation of a Brune section after an odd number of unit elements has been developed is shown in Fig. 9(a). In this figure, and also in Fig. 9(b), the S'-plane Brune section is shown in the T-section form. This is easily related, by elementary transformations [14], to the form which uses unity-coupled coils. The required impedance scaling factors have been neglected in the figures and must be supplied in the actual application. The particular form of the transformed Brune section shown in Fig. 9(a) has been given by Guillemin [14]. Other equivalent circuits are possible, but they are not presented here. The transformation of a Brune section after an even number of unit elements has been developed is shown in Fig. 9(b).

[8] In the present context L represents the characteristic impedance of a quarter-wave short-circuited transmission line, and C represents the characteristic admittance of a quarter-wave open-circuited transmission line.



(a). Transformation of a Brune section after an odd number of unit elements has been developed. (Impedance scaling has been neglected.)



(b). Transformation of a Brune section after an even number of unit elements has been developed. (Impedance scaling has been neglected.)

Fig. 9.

III. Bandwidth Formulas

The effect of the transformation $S' \rightarrow 1/S$ on bandwidth can be determined using (8). Let θ_2' and θ_1' be the upper and lower bandedges, respectively, of a given network response in the S' plane. Let the fractional bandwidth of the network be defined as

$$w' = 2\frac{\theta_2' - \theta_1'}{\theta_2' + \theta_1'} \cdot \tag{42}$$

There are two cases to consider:

Case 1: Both θ_2' and θ_1' lie between 0 and 90°.[9]
Case 2: $\theta_2' = 180° - \theta_1'$.

Case 1

Under the transformation $S' \rightarrow 1/S$, the new bandedges are

$$\theta_2 = 90° - \theta_1',$$
$$\theta_1 = 90° - \theta_2'.$$

Therefore, in the W plane the fractional bandwidth is

$$w = 2\left\{\frac{\theta_2 - \theta_1}{\theta_2 + \theta_1}\right\} = 2\left\{\frac{\theta_2' - \theta_1'}{180 - (\theta_2' + \theta_1')}\right\} \tag{43}$$

which reduces to

$$\frac{1}{w'} + \frac{1}{w} = \frac{90°}{\theta_2' - \theta_1'} \cdot \tag{44}$$

[9] An example of a Case 1 network is the short-step transformer [6].

Case 2

Under the transformation $S' \rightarrow 1/S$, the new bandedges are

$$\theta_2 = 90° + \theta_1',$$
$$\theta_1 = \theta_2' - 90°.$$

Therefore, in the W plane the fractional bandwidth is

$$w = 2\left\{\frac{\theta_2 - \theta_1}{\theta_2 + \theta_1}\right\} = 2\left\{\frac{180 - (\theta_2' - \theta_1')}{\theta_1' + \theta_2'}\right\}, \tag{45}$$

which reduces to

$$w + w' = 2. \tag{46}$$

IV. Four Theorems for Symmetrical and Antimetrical Filters

Define a symmetrical filter as one with impedances that satisfy

$$Z'_{n+1-i} = r' Z_i', \tag{47}$$

where Z_i' is the impedance of the ith stub or unit element normalized to the generator resistance, n is the total number of stubs and unit elements in the filter, and r' is the ratio of termination to generator resistance. Define an antimetrical filter as one with impedances that satisfy

$$Z'_{n+1-i} = \frac{r'}{Z_i'} \cdot \tag{48}$$

Then the following four theorems can be stated:

Theorem 1: Under the transformation $S' \rightarrow 1/S$, a symmetrical filter having M unit elements, where M is even, goes into another symmetrical filter with respect to the transformed normalized load r. The transformed normalized load r is equal to r' if $M/2$ is even, and is equal to $1/r'$ if $M/2$ is odd.

Theorem 2: Under the transformation $S' \rightarrow 1/S$, a symmetrical filter having M unit elements, where M is odd, goes into an antimetrical filter with respect to the new normalized load r. The normalized load r satisfies

$$r = \left(Z_1' Y_2' Z_3' \cdots \frac{Z'_{M-1}}{2}\right)^4 \left(\frac{Y'_{M+1}}{2}\right)^2 \tag{49}$$

for $[(M+1)/2]$ even, and

$$r = \left(Z_1' Y_2' Z_3' \cdots \frac{Y'_{M-1}}{2}\right)^4 \left(\frac{Z'_{M+1}}{2}\right)^2 \tag{50}$$

for $[(M+1)/2]$ odd.

The notation Z_j' represents the jth unit element of the S'-plane network.

Theorem 3: Under the transformation $S' \rightarrow 1/S$, an antimetrical filter having M unit elements, where M is odd, goes into a symmetrical filter. The transformed normalized termination r is unity for all cases.

Theorem 4: Under the transformation $S' \to 1/S$, an antimetrical filter having M unit elements, where M is even, goes into another antimetrical filter with respect to the new normalized termination r. The normalized termination r satisfies

$$r = \left(Z_1' Y_2' Z_3' \cdots \frac{Z_M'}{2}\right) \frac{1}{r'} \tag{51}$$

for $M/2$ odd, and

$$r = \left(Z_1' Y_2' Z_3' \cdots \frac{Y_M'}{2}\right)^4 r' \tag{52}$$

for $M/2$ even.

It is emphasized that the preceding theorems are based only on the number of unit elements in the filter and depend in no way on the number of stubs, transformers, or resistors.

V. Applications

A. Extension of Chebyshev Transformer Tables

A potential application for the transformation $S' \to 1/S$ is the extension of Chebyshev transformer tables [12] by using Levy's tables for distributed lowpass filters [11]. Published tables of exact designs of Chebyshev transformers [12] are presently limited to four sections and impedance ratios of ≤ 100. Although the range of impedance ratios covered is probably adequate for most applications, it would be useful to have tables for larger numbers of sections. Levy's tables of distributed filters may be useful for this purpose in some instances.

For example, suppose it is required to match into a network over a 3:1 bandwidth, i.e., $w' = 1.0$. Let the input impedance of the network be 15 times the source impedance, and let it also be required that the VSWR of the match be ≤ 1.05. A 15:1 impedance mismatch ratio corresponds to a maximum insertion loss of 6.28 dB. Therefore, upon scanning the tables of distributed lowpass filters [11] for a design with a maximum insertion loss of 6.3 dB and a passband VSWR of 1.05, it is found that a five-section filter is required. This particular filter has 6.22-dB insertion loss, which corresponds to an impedance mismatch of 14.6:1. The bandwidth of this filter is $w' = 0.9$. By (46), the bandwidth of the transformed network will be $w = 1.1$, which fulfills the requirements. The impedances of the transformed network can be calculated from (22) and (23), giving

$$Z_1 = 1.193$$

$$Z_2 = (1.193)^2/(0.7481) = 1.902$$

$$Z_3 = \frac{(1.193)^2(1.507)}{(0.7481)^2} = 3.832$$

$$Z_4 = \frac{14.6}{Z_2} = 7.676$$

$$Z_5 = \frac{14.6}{Z_3} = 12.23.$$

Similar methods are also applicable to the tables of short-step transformers [6], as shown by the example given in Section II-A.

B. Development of Narrowband Directional Coupler

From time to time, technical problems arise in which narrowband directional couplers prove to be more useful than wideband directional couplers. In the following example a narrowband 3-dB (approximately) directional coupler is developed using the transformation $S' \to 1/S$. The stepped-impedance TEM directional coupler has been shown to be mathematically equivalent to the stepped-impedance filter terminated in a 1-ohm resistor [15], [16]. The reflection coefficient and the insertion loss of the stepped-impedance filter correspond to the coupling and transmission, respectively, of the directional coupler. A 3-dB directional coupler thus corresponds to a stepped-impedance filter with 3-dB insertion loss. On the other hand, under the transformation $S' \to 1/S$ a stepped-impedance filter with 3-dB insertion loss corresponds to a transformer with 5.83:1 termination ratio. The tables of transformer design [10] give the following impedance values for a three-section transformer with termination ratio 6:1 and fractional bandwidth $w' = 1.2$:

$$Z_1' = 1.58676$$

$$Z_2' = 2.4495$$

$$Z_3' = 3.78129.$$

Transforming these impedances according to (22) and (23) gives the following impedance values of the stepped-impedance filter. These impedances are also the even-mode impedances of the directional coupler [19].

$$Z_1 = 1.58676 = Z_{0-\text{even}_1}$$

$$Z_2 = 1.02789 = Z_{0-\text{even}_2}$$

$$Z_3 = 1.58676 = Z_{0-\text{even}_3}.$$

Note that an unusual feature of this 3-dB coupler design is that the tightest coupling is at the ends of the coupler, as contrasted to the middle in conventional designs.[10] The coupling of the ends is

$$\text{coupling} = 20 \log_{10} \left\{\frac{Z_1^2 - 1}{Z_1^2 + 1}\right\}$$

$$= 20 \log_{10} \left\{\frac{1.51}{3.51}\right\} = -7.30 \text{ dB}.$$

Thus a second unusual feature of this design is that the tightest coupling is less than the overall coupling of the coupler. Important also is the fact that since the tightest coupling is only -7.3 dB, this coupler should be relatively easy to construct. The coupling response is shown in Fig. 10. The peak coupling is -2.92 dB, and the 3-dB fractional bandwidth has been determined from the data to be $w = 0.233$.

[10] Since the even-mode impedances Z_3 and Z_1 are greater than Z_2.

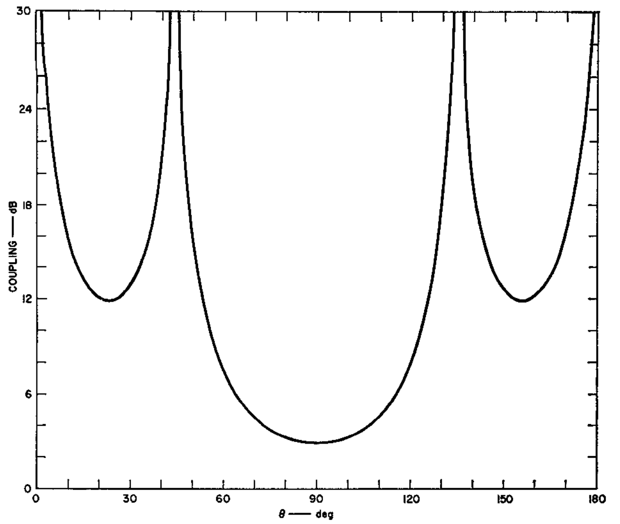


Fig. 10. Coupling response of 3-dB directional coupler designed using the transformation $S' = 1/S$.

C. Transformation of Bandstop Filters into Bandpass Filters

Perhaps the most general application of the transformation $S' \to 1/S$ will be the transformation of bandstop filters into bandpass filters (and vice versa). This application will realize a saving of 50 percent in the compilation of filter-design tables. For example, the existing tables of bandstop filters [13] are easily transformed into tables for bandpass filters. This was done for a limited number of cases in order to gain an impression of the impedance levels of the resulting bandpass filters. It was found that for a representative case (a five-stub filter with 1.2 VSWR ripple in the passband) the transformed impedances lay in the range 10 to 250 ohms over fractional bandwidths of 0.8 to 1.5. For bandwidths less than 0.8 or greater than 1.5, special methods of design (such as use of additional redundant unit elements) [17] will be required for practical realizations.

It should be pointed out that for filters having lumped-element prototypes [13], it is not necessary to use the transformation $S' \to 1/S$ on the transmission-line filter. One can apply it to the lumped-element prototype as well, utilizing Kuroda's identities to realize the transmission-line filter.[11] It is most important to note, however, that *for the classes of filters having no lumped-element prototype (such as are represented by the insertion-loss functions given in reference [17]), transformation from bandpass to bandstop (or vice versa) can be accomplished only by the techniques given in Section II or by exact synthesis methods.*

VI. Conclusions

A theoretical study has shown that the frequency transformation $S' \to 1/S$,when applied to commensurate transmission-line networks of stubs, unit elements, ideal transformers, and resistors, may be easily related to changes in the impedance values of the network parameters. In most cases, both the form and the element values of the transformed network may be written down from inspection. Because of the generality of the transformation, only a few examples could be illustrated in the present paper. However, the fundamental transformation technique applies to a wide variety of situations. The transformation will probably be most useful in reducing the number of design tables required for frequently used filter designs. However, in some cases it may also be helpful in extending existing tables of designs.

Knowledge of the transformation also provides an alternative viewpoint to various network synthesis problems, as exemplified by the design of the narrowband 3-dB directional coupler. Also, for some network synthesis problems it may prove useful to work in the W plane rather than the S' plane and then transform the resulting network. For example, retention of significant figures in an electronic digital computer may be a problem in synthesizing narrowband bandpass filters, but it may not be a problem in synthesizing wideband bandstop filters. Thus the latter could be accomplished and the network transformed thereafter.

[11] However, in many cases, it will probably be faster to use the computational algorithm [13] in designing the corresponding bandstop filters, or the tables [13], and then convert the design to a bandpass filter by the transformation $S' \to 1/S$.

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